Corrigé type Examen 2025 (22 pts)

Exo 1 (7 pts)

1–

$$E_s \leq \frac{(b-a)^5}{180 n^4} \max_{[a,b]} |f^{(4)}(x)|$$

we have $f^{(4)}(x) = (16x^4 + 48x^2 + 12)e^{x^2}$

and $f^{(5)}(x) = (32x^5 + 16x^3 + 120x)e^{x^2} > 0, \forall x \in [0,1]$

the $\max_{[0,1]} |f^{(4)}(x)| = f^{(4)} = f^{(4)}(1) = 76e = 206.5894$ (1)

then:

$\cancel{E_s \leq \frac{1}{180 n^4}}$

$$n \geq \sqrt[4]{\frac{(b-a)^5 \max_{[a,b]} |f^{(4)}(x)|}{180 \times \varepsilon}}$$ 0.5

$$n \geq \left(\frac{1 \times 206.5894}{180 \times 10^{-3}}\right)^{\frac{1}{4}} = 5.8204$$

then: $n = 6$ (1)

Ex 03 (8 pts)

$$(P)\quad \begin{cases} \dfrac{dy}{dt} + \cos(t)\, y = -\cos(t), & y \in [0,\pi] \\ y(0) = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} f^*(t) = -\cos(t)\left(y(t)+1\right) \\ y(0) = 0 \end{cases}$$

1- Euler Method: $\begin{cases} y_0 = y(a) \\ y_{i+1} = y_i + h \times f(t_i, y_i) \end{cases}$ (0,5)

$t_i = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

then:

$y(\frac{\pi}{4}) \simeq y_1 = y_0 + \frac{\pi}{4} f(0, y_0) = \boxed{-\frac{\pi}{4}}$ (0,5)

$y(\frac{\pi}{2}) \simeq y_2 = y_1 + \frac{\pi}{4} f(t_1, y_1) = -\frac{\pi}{4} + \frac{\pi}{4} f(\frac{\pi}{4}, y_1) = \boxed{-0.9045}$ (0,5)

$y(\frac{3\pi}{4}) \simeq y_3 = y_2 + \frac{\pi}{4} \times f(t_2, y_2) = \boxed{-0.9045}$ (0,5)

$y(\pi) \simeq y_4 = y_3 + \frac{\pi}{4} f(t_3, y_3) = \boxed{-0.8514}$ (0,5)

hence $\boxed{y(\pi) \simeq y_4 = -0.8514}$ (0,5) then for $k = 1$:

$$\begin{cases} x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{f(3)}{f'(3)} = 2.333 \\ \Delta_1 = \dfrac{\Pi}{2m}(x_1 - x_0)^2 = 0.1111 \end{cases}$$

0,5

• For $k = 2$

$$\begin{cases} x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.2380 \\ \Delta_2 = \dfrac{\Pi}{2m}(x_2 - x_1)^2 = 2.2705 \end{cases}$$

0,5 ?!

For $k = 3$

$$\begin{cases} x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.236 \\ \Delta_3 = \dfrac{\Pi}{2m}(x_3 - x_2)^2 = 10^{-6} < \varepsilon \end{cases}$$

0,5

hence $\sqrt{5} \simeq 2.236 \pm 10^{-6}$ 1

4- $n \geq \dfrac{\ln\left(\frac{b-a}{2\varepsilon}\right)}{\ln(2)} = \dfrac{\ln\left(\frac{1}{2\times10^{-4}}\right)}{\ln(2)} = 12.2877$ 0,5

then $n = 13$ 1

the evaluation points are: (with $h = \frac{1}{6}$).

x_i	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x_i)$	1	1.1942	1.4508	1.784	2.2262	2.8359	3.7182

then

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left[f(a) + f(b) + 2\sum_{i=1}^{k-1} f(x_{2i}) + 4\sum_{i=1}^{k} f(x_{2i-1}) \right]$$

with $k = \frac{n}{2}$ 0,5

hence:

$$I = \frac{1}{18}\left[1 + 3.7182 + 2(1.4508 + 2.2262) + 4(1.1948 + 1.7840 + 2.8359 + 3.7182) \right]$$ 1

$$\boxed{\simeq 1.9628}$$ 0,5

hence $I \simeq 1.9628$.

2-

$$n \geq \sqrt{\frac{(b-a)^3 \max_{[a,b]} |f''(x)|}{12 \cdot \varepsilon}}$$ 0,5

we have: $f''(x) = (2+4x^2)e^{x^2}$ with $f^{(3)}(x) = (8x^3+12x)e^{x^2} > 0$ $\forall x \in [0,1]$

then: $\max_{[0,1]} f^{(2)}(x) = f^{(2)}(1) = 6e = 16.3096$ 1

then:

$$n \geq \sqrt{\frac{1}{12\times 10^{-3}} \, 16.3096} = 36,8664$$

then $\boxed{n = 37}$ 1

Ex02 (7pts)

1- the intervalle where we estimate $\sqrt{5}$ is: $\boxed{[2,3]}$ 0,5

2- the expression of the function f is:

$$\boxed{f(x) = x^2 - 5}$$ 1

3- the iterative Schema of Newton-Raphson is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$ 0,5

Stopping criterion is: $\Delta_k = \frac{M}{2m}(x_k - x_{k-1})^2 \leq 10^{-4}$

with: $M = \max_{[a,b]} |f''(x)| = 4$ and $m = \min_{[2,3]} |f'(x)| = 4$

0,5 0,5

$k=4$:

$K_1 = 0,3575$ and $K_2 = 0,7865$

then $\boxed{y(\pi) \simeq y_4 = -0,045}$ 0,5

hence $y(\pi) \simeq -0,045$ 0,5

2- : $y'(t) = -\cos(t) e^{-\sin(t)}$ 1

then $y'(t) + \cos(t) y(t) = -\cos(t) e^{-\sin(t)} + \cos(t) e^{-\sin} - \cos(t)$

$= -\cos(t)$

and $y(0) = 0$, hence it a solution of (P).

3- we have $y(\pi) = 0$ (exact solution)

- $Err_{Euler} = |-0.8515 - 0| = 0.8515$
- $Err_{RK_2} = |-0,045 - 0| = 0.045$

1

we remak that $Err_{Euler} << Err_{RK_2}$, the

the RK_2 method is more precise than Euler.

* RK 2

R.K$_2$ $\begin{cases} y_0 = y(t_0) \\ y_{i+1} = y_i + \frac{h}{2}(K_1 + K_2) \\ \text{with } K_1 = f(t_i, y_i) \\ \quad\quad K_2 = f(t_i + h, y_i + h.K_1) \end{cases}$ (0,5)

$$f(t,y) = -\cos(t)(y+1)$$

k = 1

$K_1 = -1$ and ~~K_2~~ $K_2 = -0,1517$

then: $\boxed{y(\frac{\pi}{4}) \simeq y_1 = -0,4522}$ (0,5)

k = 2 $K_1 = -0,3873$ and $K_2 = 0$.

then $\boxed{y(\frac{\pi}{2}) \simeq y_2 = -0,6042}$ (0,5)

k = 3 : $K_1 = 0$ and $K_2 = 0.2798$

then $\boxed{y(\frac{3\pi}{4}) \simeq y_3 = -0,4943}$ (0,5)