الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي

Department of Mathematics
Faculty of Sciences
Setif 1 University-Ferhat ABBAS

Integral transforms in Lp spaces

Thursday,
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Duration: 1h 30m

Exam

Exercise N° 01: (05 Points)

Let $p, q, r \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

Show that if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

where Ω denotes an open set of IR .

Exercise N° 02: (09 Points)

We define the Fourier transform of the function $f \in L^1(IR)$ by the following formula

$$F(f(x))(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \varpi x} \, dx, \omega \in IR.$$

1. Compute the Fourier transforms of

$$(\varphi_a f)(x) = f(x-a),\ a \in IR \text{ and } (h_\lambda f)(x) = f(\lambda x), \lambda \in IR.$$

2. Let the functions be defined on IR by

$$f(x) = \frac{1}{1+x^2},\ \ g(x) = \frac{1}{2} \times \frac{1}{1+x^2} + f(-x),\ k(x) = \frac{4}{5-2x+x^2}$$

Knowing that $\hat{f}(\omega) = \pi e^{-2\pi|\omega|}$, determine $\hat{g}(\omega)$ and $\hat{k}(\omega)$.

Exercise N° 03: (06 Points)

Use the Laplace transform to solve the following differential equation:

$$y''(t) - y(t) = 3e^{-2t} + t + 1 \qquad (1)$$

with the initial conditions

$$y(0) = y'(0) = 0 \qquad (2)$$

Good luck **Dr. Ali KHALOUTA**

Typical Correction of Exam « Integral transforms in Lp spaces »

Exercise N° 01 (05 Points) :

Let $p, q, r \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

We show that if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\| fg\|_{L^r} \leq \| f\|_{L^p} \|g\|_{L^q}$$

where Ω denotes an open set of IR.

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For this, we have

$$\| fg\|_{L^r}^r = \int_\Omega | fg|^r \, dx = \int_\Omega | f|^r |g|^r \, dx$$

On the other hand we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Rightarrow \frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$$

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According to Hölder's inequality, we have

$$\begin{aligned} \| fg\|_{L^r}^r &= \int_\Omega | f|^r |g|^r \, dx \\ &\leq \left(\int_\Omega \left(| f|^r\right)^{\frac{p}{r}} dx \right)^{\frac{r}{p}} \left(\int_\Omega \left(|g|^r\right)^{\frac{q}{r}} dx \right)^{\frac{r}{q}} \\ &\leq \left(\left(\int_\Omega \left(| f|\right)^p dx\right)^{\frac{1}{p}} \right)^r \left(\left(\int_\Omega \left(|g|\right)^q dx\right)^{\frac{1}{q}} \right)^r \\ &\leq \| f\|_{L^p}^r \|g\|_{L^q}^r \end{aligned}$$

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Which implies that

$$\| fg\|_{L^r} \leq \| f\|_{L^p} \|g\|_{L^q}$$

Since $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, the we have $\| f\|_{L^p} < \infty$ and $\|g\|_{L^q} < \infty$, which implies that $\| fg\|_{L^r} < \infty$, so $fg \in L^r(\Omega)$.

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Exercise Nº 02 (09 Points) :

1. We compute the Fourier transforms of

$$(\varphi_a f)(x) = f(x-a),\ a \in IR \text{ and } (h_\lambda f)(x) = f(\lambda x), \lambda \in IR.$$

where

$$F(f(x))(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x}\, dx, \omega \in IR.$$

i) With the change of variable $y = x - a$, we find

$$F((\varphi_a f)(x))(\omega) = F(f(x-a))(\omega) = \int_{-\infty}^{+\infty} f(x-a) e^{-2\pi i \omega x}\, dx$$

$$= \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega (y+a)}\, dy$$

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$$= e^{-2\pi i \omega a} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega y}\, dy$$

$$= e^{-2\pi i \omega a} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega y}\, dy$$

$$= e^{-2\pi i \omega a} \hat{f}(\omega)$$

ii) By definition ,we have

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \int_{-\infty}^{+\infty} f(\lambda x) e^{-2\pi i \omega x}\, dx$$

There are two cases

- If $\lambda > 0,$ we put $y = \lambda x \Rightarrow dy = \lambda dx$, then

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega \frac{y}{\lambda}} \frac{dy}{\lambda}$$

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$$= \frac{1}{\lambda} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i y \left(\frac{\omega}{\lambda}\right)}\, dy$$

$$= \frac{1}{\lambda} \hat{f}\left(\frac{\omega}{\lambda}\right)$$

- If $\lambda < 0,$ then

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \int_{+\infty}^{-\infty} f(y) e^{-2\pi i \omega \frac{y}{\lambda}} \frac{dy}{\lambda}$$

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$$= -\frac{1}{\lambda} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i y \left(\frac{\omega}{\lambda}\right)}\, dy$$

$$= -\frac{1}{\lambda} \hat{f}\left(\frac{\omega}{\lambda}\right)$$

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Finally, we have

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\omega}{\lambda}\right)$$

2. Let the functions be defined on IR by

$$f(x)=\frac{1}{1+x^2},\ \ g(x)=\frac{1}{2}\times\frac{1}{1+x^2}+f(-x),\ k(x)=\frac{1}{x^2-2x+5}$$

Knowing that $\hat{f}(\omega)=\pi e^{-2\pi|\omega|}$, we determine $\hat{g}(\omega)$ and $\hat{k}(\omega)$.

i) For the first function $g(x)$, we have

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$$g(x)=\frac{1}{2}f(x)+f(-x)$$

Which implies that

$$\hat{g}(\omega)=F\left(\frac{1}{2}f(x)+f(-x)\right)(\omega)$$

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$$=\frac{1}{2}F\big(f(x)\big)(\omega)+F\big(f(-x)\big)(\omega)$$

$$=\frac{1}{2}\hat{f}(\omega)+\hat{f}(-\omega)$$

$$=\frac{1}{2}\pi e^{-2\pi|\omega|}+\pi e^{-2\pi|-\omega|}$$

$$=\frac{3}{2}\pi e^{-2\pi|\omega|}$$

ii) For the second function $k(x)$, we have

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$$k(x)=\frac{4}{5-2x+x^2}=\frac{1}{1+\left(\frac{x-1}{2}\right)^2}$$

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Which implies that

$$k(x)=\left(h_{\frac{1}{2}}(\varphi_1 f)\right)(x)=f\left(\frac{x-1}{2}\right)=f\left(\frac{1}{2}(x-1)\right)$$

Then

$$\hat{k}(\omega)=F\left(\left(h_{\frac{1}{2}}(\varphi_1 f)\right)(x)\right)(\omega)=2(\hat{\varphi}_1 f)(2\omega)$$

$$=2e^{-2\pi i(2\omega)}\hat{f}(2\omega)$$

$$=2e^{-4\pi i\omega}\pi e^{-2\pi|2\omega|}$$

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Exercise N° 03 (06 Points) :

We use the Laplace transform to solve the following differential equation:

$$y''(t) - y(t) = 3e^{-2t} + t + 1 \qquad (1)$$

with the initial conditions

$$y(0) = y'(0) = 0 \qquad (2)$$

Applying the Laplace transform to both sides of equation (1) and using the linearity property, we obtain:

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$$L\left(y''(t)\right) - L\left(y(t)\right) = 3L\left(e^{-2t}\right) + L(t) + L(1)$$

Using the Laplace transform of the second derivative, we have:

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$$p^2 L\left(y(t)\right) - py(0) - y'(0) - L\left(y(t)\right) = \frac{3}{p+2} + \frac{1}{p^2} + \frac{1}{p}$$

By substituting the initial conditions (2), we find

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$$p^2 L\left(y(t)\right) - L\left(y(t)\right) = \frac{3}{p+2} + \frac{1}{p^2} + \frac{1}{p}$$

By a simple calculation, we obtain

$$\left(p^2 - 1\right) L\left(y(t)\right) = \frac{4p^2 + 3p + 2}{p^2(p+2)}$$

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Which implies that

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$$L\left(y(t)\right) = \frac{4p^2 + 3p + 2}{p^2(p+2)(p^2-1)}$$

$$= -\frac{1}{p} - \frac{1}{p^2} + \frac{1}{p+2} + \frac{3}{2}\left(\frac{1}{p-1}\right) - \frac{3}{2}\left(\frac{1}{p+1}\right)$$

Applying the inverse Laplace transform, we find the solution to equations (1) and (2) as follows

$$y(t) = -1 - t + e^{-2t} + \frac{3}{2}e^{t} - \frac{3}{2}e^{-t}$$

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Dr. Ali KHALOUTA