



Exam

Exercise N° 01: (05 Points)

Let $p, q, r \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

Show that if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

where Ω denotes an open set of \mathbb{R} .

Exercise N° 02: (09 Points)

We define the Fourier transform of the function $f \in L^1(\mathbb{R})$ by the following formula

$$F(f(x))(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} dx, \omega \in \mathbb{R}.$$

1. Compute the Fourier transforms of

$$(\varphi_a f)(x) = f(x-a), a \in \mathbb{R} \text{ and } (h_\lambda f)(x) = f(\lambda x), \lambda \in \mathbb{R}.$$

2. Let the functions be defined on \mathbb{R} by

$$f(x) = \frac{1}{1+x^2}, g(x) = \frac{1}{2} \times \frac{1}{1+x^2} + f(-x), k(x) = \frac{4}{5-2x+x^2}$$

Knowing that $\hat{f}(\omega) = \pi e^{-2\pi|\omega|}$, determine $\hat{g}(\omega)$ and $\hat{k}(\omega)$.

Exercise N° 03: (06 Points)

Use the Laplace transform to solve the following differential equation:

$$y''(t) - y(t) = 3e^{-2t} + t + 1 \quad (1)$$

with the initial conditions

$$y(0) = y'(0) = 0 \quad (2)$$

Good luck

Dr. Ali KHALOUTA

Typical Correction of Exam « Integral transforms in L^p spaces »

Exercise N° 01 (05 Points) :

Let $p, q, r \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

We show that if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

where Ω denotes an open set of \mathbb{R} .

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For this, we have

$$\|fg\|_{L^r}^r = \int_{\Omega} |fg|^r dx = \int_{\Omega} |f|^r |g|^r dx$$

On the other hand we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Rightarrow \frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$$

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According to Hölder's inequality, we have

$$\begin{aligned} \|fg\|_{L^r}^r &= \int_{\Omega} |f|^r |g|^r dx \\ &\leq \left(\int_{\Omega} (|f|^r)^{\frac{p}{r}} dx \right)^{\frac{r}{p}} \left(\int_{\Omega} (|g|^r)^{\frac{q}{r}} dx \right)^{\frac{r}{q}} \\ &\leq \left(\left(\int_{\Omega} (|f|^p) dx \right)^{\frac{1}{p}} \right)^r \left(\left(\int_{\Omega} (|g|^q) dx \right)^{\frac{1}{q}} \right)^r \\ &\leq \|f\|_{L^p}^r \|g\|_{L^q}^r \end{aligned}$$

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Which implies that

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

Since $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then we have $\|f\|_{L^p} < \infty$ and $\|g\|_{L^q} < \infty$, which implies that $\|fg\|_{L^r} < \infty$, so $fg \in L^r(\Omega)$.

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Exercise N° 02 (09 Points) :

1. We compute the Fourier transforms of

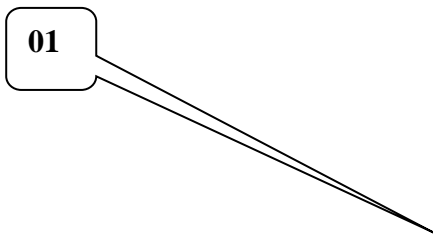
$$(\varphi_a f)(x) = f(x-a), a \in \mathbb{R} \text{ and } (h_\lambda f)(x) = f(\lambda x), \lambda \in \mathbb{R}.$$

where

$$F(f(x))(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} dx, \omega \in \mathbb{R}.$$

i) With the change of variable $y = x - a$, we find

$$\begin{aligned} F((\varphi_a f)(x))(\omega) &= F(f(x-a))(\omega) = \int_{-\infty}^{+\infty} f(x-a) e^{-2\pi i \omega x} dx \\ &= \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega (y+a)} dy \\ &= e^{-2\pi i \omega a} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega y} dy \\ &= e^{-2\pi i \omega a} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega y} dy \\ &= e^{-2\pi i \omega a} \hat{f}(\omega) \end{aligned}$$



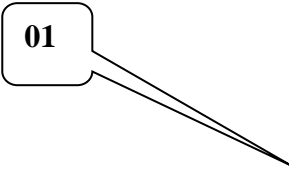
ii) By definition, we have

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \int_{-\infty}^{+\infty} f(\lambda x) e^{-2\pi i \omega x} dx$$

There are two cases

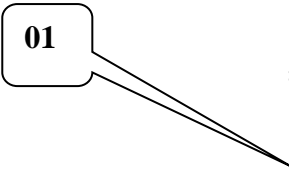
- If $\lambda > 0$, we put $y = \lambda x \Rightarrow dy = \lambda dx$, then

$$\begin{aligned} F((h_\lambda f)(x))(\omega) &= F(f(\lambda x))(\omega) = \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \omega \frac{y}{\lambda}} \frac{dy}{\lambda} \\ &= \frac{1}{\lambda} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i y \left(\frac{\omega}{\lambda}\right)} dy \\ &= \frac{1}{\lambda} \hat{f}\left(\frac{\omega}{\lambda}\right) \end{aligned}$$



- If $\lambda < 0$, then

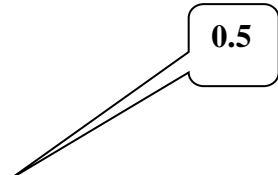
$$\begin{aligned} F((h_\lambda f)(x))(\omega) &= F(f(\lambda x))(\omega) = \int_{+\infty}^{-\infty} f(y) e^{-2\pi i \omega \frac{y}{\lambda}} \frac{dy}{\lambda} \\ &= -\frac{1}{\lambda} \int_{-\infty}^{+\infty} f(y) e^{-2\pi i y \left(\frac{\omega}{\lambda}\right)} dy \\ &= -\frac{1}{\lambda} \hat{f}\left(\frac{\omega}{\lambda}\right) \end{aligned}$$



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Finally, we have

$$F((h_\lambda f)(x))(\omega) = F(f(\lambda x))(\omega) = \frac{1}{|\lambda|} \hat{f}\left(\frac{\omega}{\lambda}\right)$$



2. Let the functions be defined on \mathbb{R} by

$$f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{1}{2} \times \frac{1}{1+x^2} + f(-x), \quad k(x) = \frac{1}{x^2 - 2x + 5}$$

Knowing that $\hat{f}(\omega) = \pi e^{-2\pi|\omega|}$, we determine $\hat{g}(\omega)$ and $\hat{k}(\omega)$.

i) For the first function $g(x)$, we have

$$g(x) = \frac{1}{2} f(x) + f(-x)$$

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Which implies that

$$\begin{aligned} \hat{g}(\omega) &= F\left(\frac{1}{2} f(x) + f(-x)\right)(\omega) \\ &= \frac{1}{2} F(f(x))(\omega) + F(f(-x))(\omega) \\ &= \frac{1}{2} \hat{f}(\omega) + \hat{f}(-\omega) \\ &= \frac{1}{2} \pi e^{-2\pi|\omega|} + \pi e^{-2\pi|-\omega|} \\ &= \frac{3}{2} \pi e^{-2\pi|\omega|} \end{aligned}$$

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ii) For the second function $k(x)$, we have

$$k(x) = \frac{4}{5-2x+x^2} = \frac{1}{1+\left(\frac{x-1}{2}\right)^2}$$

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Which implies that

$$k(x) = \left(h_{\frac{1}{2}}(\varphi_1 f)\right)(x) = f\left(\frac{x-1}{2}\right) = f\left(\frac{1}{2}(x-1)\right)$$

Then

$$\begin{aligned} \hat{k}(\omega) &= F\left(\left(h_{\frac{1}{2}}(\varphi_1 f)\right)(x)\right)(\omega) = 2(\hat{\varphi}_1 f)(2\omega) \\ &= 2e^{-2\pi i(2\omega)} \hat{f}(2\omega) \\ &= 2e^{-4\pi i\omega} \pi e^{-2\pi|2\omega|} \end{aligned}$$

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Exercise N° 03 (06 Points) :

We use the Laplace transform to solve the following differential equation:

$$y''(t) - y(t) = 3e^{-2t} + t + 1 \quad (1)$$

with the initial conditions

$$y(0) = y'(0) = 0 \quad (2)$$

Applying the Laplace transform to both sides of equation (1) and using the linearity property, we obtain:

$$L(y''(t)) - L(y(t)) = 3L(e^{-2t}) + L(t) + L(1)$$

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Using the Laplace transform of the second derivative, we have:

$$p^2 L(y(t)) - py(0) - y'(0) - L(y(t)) = \frac{3}{p+2} + \frac{1}{p^2} + \frac{1}{p}$$

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By substituting the initial conditions (2), we find

$$p^2 L(y(t)) - L(y(t)) = \frac{3}{p+2} + \frac{1}{p^2} + \frac{1}{p}$$

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By a simple calculation, we obtain

$$(p^2 - 1)L(y(t)) = \frac{4p^2 + 3p + 2}{p^2(p+2)}$$

Which implies that

$$L(y(t)) = \frac{4p^2 + 3p + 2}{p^2(p+2)(p^2 - 1)}$$

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$$= -\frac{1}{p} - \frac{1}{p^2} + \frac{1}{p+2} + \frac{3}{2} \left(\frac{1}{p-1} \right) - \frac{3}{2} \left(\frac{1}{p+1} \right)$$

Applying the inverse Laplace transform, we find the solution to equations (1) and (2) as follows

$$y(t) = -1 - t + e^{-2t} + \frac{3}{2}e^t - \frac{3}{2}e^{-t}$$

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