الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي و البحث العلمي

Integral transforms in Lp spaces

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Duration: 1h 30m

Catch-up exam

Exercise N° 01: (05 Points)

Let $(\Omega\,,\,M\,,\,\mu)$ be a finite measure space $(\mu(\Omega)<\infty)$

1- Give the definition of the Lebesgue space norm $L^{\infty}(\Omega)$

2- Show that: If $f\in L^{\infty}(\Omega)$ then $|f(x)|\leq \|f\|_{\infty}\quad a.e\quad on\quad \Omega$

3- Let $p\in[1,+\infty[$. Show that

$$\forall\, p\geq 1\,,\; L^{\infty}\left(]1,2[\right)\subset L^{p}\left(]1,2[\right)$$

Exercise N° 02: (06 Points)

1- Find the Fourier transform of the following function :

$$f(x)=\begin{cases}1 & si\quad |x|\leq a\\ 0 & si\quad |x|>a\end{cases}$$

2- Using the inverse Fourier transform, compute the integral

$$\int_{-\infty}^{+\infty}\frac{\cos\,(2\pi\omega\,x)\sin\,(2\pi\omega\,a)}{\omega}\,d\,\omega$$

3- Deduce the value of the integral

$$\int_{0}^{+\infty}\frac{\sin\,(x)}{x}\,d\,x$$

Exercise N° 03: (09 Points)

1-_Compute the Laplace transform of $e^{-\alpha t}\,,\,\alpha\in C$

2- Deduce the Laplace transform of $\cos\,(a\,t)$ and $\sin\,(a\,t),\; a\in IR$

3- Use the Laplace transform to solve the following differential equation:

$$y''(t)-y(t)=3e^{-2t}+t+1 \qquad (1)$$

with the initial conditions

$$y(0)=y'(0)=\;0 \qquad (2)$$

Good luck **Dr. Ali KHALOUTA**

Typical Correction of Exam « Integral transforms in Lp spaces »

Exercise N° 01 (05 Points) :

Let $(\Omega\,,\,M\,,\,\mu)$ be a finite measure space $(\mu(\Omega)<\infty)$.

1- The definition of the Lebesgue space norm $L^{\infty}(\Omega)$ is : **0.5**

$$\|f\|_{\infty}=\inf\left\{C:|f(x)|\le C\ a.e\ \ on\ \ \Omega\right\}.$$

2- We show that : if $f\in L^{\infty}(\Omega)$ then $|f(x)|\le\|f\|_{\infty}\ \ a.e\ \ on\ \ \Omega$.

Indeed, there is a sequence C_n such as $C_n\to\|f\|_{\infty}$ and for every n we have $|f(x)|\le C_n$ $a.e\ \ on\ \ \Omega$. So $|f(x)|\le C_n$ for all $x\in\Omega\,/\,\mathrm{E_n}$ with E_n negligible. **01**

We put $E=\bigcup_{n}E_n$ so that E is negligible, and we have $|f(x)|\le C_n$ for all n and for all $x\in\Omega\,/\,E$. **01**

0.5

Therefore $|f(x)|\le\|f\|_{\infty}$ for all $x\in\Omega\,/\,E$.

3- Let $p\in[1,+\infty[$. We show that

$$\forall\, p\ge 1\,,\ L^{\infty}\left(]1,2[\right)\subset L^{p}\left(]1,2[\right)$$

0.5

Let $f\in L^{\infty}\left(]1,2[\right)$ then $|f(x)|\le\|f\|_{\infty}\ \ a.e\ \ on\ \ ]1,2[$. **01**

$$\int_1^2|f(x)|^{\,p}\,dx\le\int_1^2\|f\|_{\infty}^{p}\,dx=\|f\|_{\infty}^{p}\int_1^2 1dx$$

0.5

$$=\|f\|_{\infty}^{p}.1<\infty\ \ (Because\ \ \|f\|_{\infty}<\infty)$$

Then $f\in L^{p}\left(]1,2[\right)$

Exercise N° 02 (06 Points) :

1- We find the Fourier transform of the following function

$$f(x)=\begin{cases}1\ \ si & |x|\le a\\ 0\ \ si & |x|>a\end{cases}$$

$$\hat{f}(\omega)=F(f(x))(\omega)=\int_{-\infty}^{+\infty}f(x)e^{-2\pi i\,\omega x}\,dx=\int_{-a}^{a}e^{-2\pi i\,\omega x}\,dx$$

0.5 **0.5**

$$=\left[-\frac{1}{2\pi i\,\omega}e^{-2\pi i\,\omega x}\right]_{-a}^{a}$$

0.5

$$=\frac{1}{\pi\,\omega}\left(\frac{e^{2\pi i\,\omega a}-e^{-2\pi i\,\omega a}}{2i}\right)$$

$$=\frac{\sin(2\pi\,\omega\,a)}{\pi\,\omega}$$

0.5

2) We use the inverse Fourier transform to calculate the integral

$$\int_{-\infty}^{+\infty} \frac{\cos\ (2\pi\omega\, x)\sin\,(2\pi\omega\, a)}{\omega} d\,\omega$$

For this, we have

0.5

$$F^{-1}(\hat{f}(\omega)) = f(x) \Leftrightarrow \int_{-\infty}^{+\infty} e^{2\pi i\,\omega x}\ \hat{f}(\omega)d\omega = f(x)$$

0.5

$$\Leftrightarrow \int_{-\infty}^{+\infty} \left(\cos(2\pi\,\omega x) + i\sin(\ 2\pi\,\omega x\ )\right) \frac{\sin\,(2\pi\,\omega\ a)}{\pi\ \omega}\ d\omega = \begin{cases} 1 & si \quad |x| \le a \\ 0 & si \quad |x| > a \end{cases}$$

0.5

$$\Leftrightarrow \int_{-\infty}^{+\infty} \cos(2\pi\,\omega x)\ \frac{\sin\,(2\pi\,\omega\ a)}{\pi\ \omega}\ d\omega + i\int_{-\infty}^{+\infty} \sin(\ 2\pi\,\omega x)\ \frac{\sin\,(2\pi\,\omega\ a)}{\pi\ \omega}\ d\omega$$

$$= \begin{cases} 1 & si \quad |x| \le a \\ 0 & si \quad |x| > a \end{cases}$$

0.5

0.5

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos(2\pi\,\omega x)\ \sin\,(2\pi\,\omega\ a)}{\omega}\ d\omega = \begin{cases} \pi & si \quad |x| \le a \\ 0 & si \quad |x| > a \end{cases}$$

3) For $x = 0$ and $a = \dfrac{1}{2\pi}$, we have

$$\int_{-\infty}^{+\infty} \frac{\sin\,(\pi\omega)}{\omega} d\,\omega = \pi$$

0.5

$$\Rightarrow 2\int_{0}^{+\infty} \frac{\sin\,(\pi\omega)}{\omega} d\,\omega = \pi$$

0.5

$$\Rightarrow \int_{0}^{+\infty} \frac{\sin\,(\pi\omega)}{\omega} d\,\omega = \frac{\pi}{2}$$

0.5

Finally

$$\int_{0}^{+\infty} \frac{\sin\,(\pi\, x)}{x} d\,x = \frac{\pi}{2}$$

<u>Exercise N° 03 (09 Points):</u>

1- We compute the Laplace transform of $e^{-\alpha t}$, $\alpha \in C$

01

For this, we have

$$L\left(e^{-\alpha t}\right)(p) = \int_{0}^{+\infty} e^{-\alpha t}\, e^{-pt}\, dt = \int_{0}^{+\infty} e^{-(\alpha+p)t}\ dt = -\frac{1}{p+\alpha}\left[e^{-(p+\alpha)t}\right]_{0}^{+\infty}$$

Let $p=a+ib$ and $\alpha=c+id \Rightarrow p+\alpha=(a+c)+i(b+d)$ **0.5**

$$\Rightarrow \lim_{t\to+\infty} e^{-(p+\alpha)t}=0 \text{ if } \operatorname{Re}(p)+\operatorname{Re}(\alpha)>0$$ **01**

0.5

$$\Rightarrow L\left(e^{-\alpha t}\right)(p)=\frac{1}{p+\alpha} \quad \text{if} \quad \operatorname{Re}(p)+\operatorname{Re}(\alpha)>0$$

2- We deduce the Laplace transform of $\cos(at)$ and $\sin(at)$, $a\in IR$

$$L\left(\cos(at)\right)(p)=L\left(\frac{e^{iat}+e^{-iat}}{2}\right)(p)=\frac{1}{2}\left(L\left(e^{iat}\right)(p)+L\left(e^{-iat}\right)(p)\right)$$

$$=\frac{1}{2}\left(\frac{1}{p-ia}+\frac{1}{p+ia}\right)$$

01

$$=\frac{p}{p^2+a^2}, \operatorname{Re}(p)>0$$

And

$$L\left(\sin(at)\right)(p)=L\left(\frac{e^{iat}+e^{-iat}}{2i}\right)(p)=\frac{1}{2i}\left(L\left(e^{iat}\right)(p)-L\left(e^{-iat}\right)(p)\right)$$

$$=\frac{1}{2i}\left(\frac{1}{p-ia}-\frac{1}{p+ia}\right)$$

01

$$=\frac{a}{p^2+a^2}, \operatorname{Re}(p)>0$$

2- We use the Laplace transform to solve the following differential equation:

$$y''(t)-y(t)=3e^{-2t}+t+1 \qquad (1)$$

with the initial conditions

$$y(0)=y'(0)=0 \qquad (2)$$

Applying the Laplace transform to both sides of equation (1) and using the linearity property, we obtain:

0.5

$$L\left(y''(t)\right)-L\left(y(t)\right)=3L\left(e^{-2t}\right)+L(t)+L(1)$$

Using the Laplace transform of the second derivative, we have:

0.5

$$p^2L\left(y(t)\right)-py(0)-y'(0)-L\left(y(t)\right)=\frac{3}{p+2}+\frac{1}{p^2}+\frac{1}{p}$$

By substituting the initial conditions (2), we find

$$p^2 L\left(y(t)\right) - L\left(y(t)\right) = \frac{3}{p+2} + \frac{1}{p^2} + \frac{1}{p}$$

0.5

By a simple calculation, we obtain

$$\left(p^2 - 1\right) L\left(y(t)\right) = \frac{4p^2 + 3p + 2}{p^2\left(p+2\right)}$$

0.5

Which implies that

$$L\left(y(t)\right) = \frac{4p^2 + 3p + 2}{p^2\left(p+2\right)\left(p^2 - 1\right)}$$

$$= -\frac{1}{p} - \frac{1}{p^2} + \frac{1}{p+2} + \frac{3}{2}\left(\frac{1}{p-1}\right) - \frac{3}{2}\left(\frac{1}{p+1}\right)$$

01

Applying the inverse Laplace transform, we find the solution to equations (1) and (2) as follows

$$y(t) = -1 - t + e^{-2t} + \frac{3}{2}e^{t} - \frac{3}{2}e^{-t}$$

01

Dr. Ali KHALOUTA