Chapter 4

Existence and uniqueness of the solution for FDE

In this chapter, we recall fundamental notions and results of the theory of functional analysis (Banach's contraction principle, equicontinuity, Schauder's theorem, Arzela-Ascoli theorem, etc.). We will then address the question of existence and uniqueness of the solution for the boundary problem of fractional differential equation (FDE).

4.1 Some fixed point theorems

Definition 4.1.1 *Let (E, d) be a complete metric space and $F : E \to E$ a continuous application.*

i) We say that $u \in E$ is a fixed point of F if $f(u) = u$.

ii) We say that F is contracting if it is Lipschitz with ratio $0 < L < 1$, i.e. if there exists $0 < L < 1$, such that

$$\forall u, v \in E : d(F(u), F(v)) \leq Ld(u, v), 0 < L < 1.$$

Definition 4.1.2 ***(Completely continues)***

Definition 4.1.3 *Let X and Y be two Banach spaces and $F : X \to Y$ be a map, defined from X to values in Y. We say that F is completely continuous if it is continuous and* *transforms everything bounded in X into a relatively compact set in Y, F is said to be compact if $F(X)$ is relatively compact in Y.*

Theorem 4.1.1 ***(Arzela-Ascoli)***

Let A be a subset of $C(J,E)$, A is relatively compact in $C(J,E)$ if and only if the following conditions are verified

i) The set A is bounded. i.e there exists a constant $K > 0$ such that

$$\|f(x)\| \leq K \text{ for all } x \in J \text{ and } f \in A.$$

ii) The set A is equicontinuous. i.e for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|t_1 - t_2\| < \delta \Longrightarrow \|f(t_1) - f(t_2)\| < \varepsilon \text{ for all } t_1, t_2 \in J \text{ and } f \in A.$$

iii) For all $x \in J$ the set $\{f(x) : f \in A\} \subset E$ is relatively compact.

Theorem 4.1.2 ***(Banach)***

Let X be a Banach space and a contracting operator $F : X \longrightarrow X$. Then F admits a unique fixed point. i.e $\exists u \in X$ such that $Fu = u$.

The second fixed point theorem that we are going to state is that of Schauder.

Theorem 4.1.3 ***(Schauder)***

Let (E,d) be a complete metric space and X be a convex and closed part of E; and let $F : X \longrightarrow X$ a map such that the set $\{Fu : u \in X\}$ is relatively compact in E. Then F has at least one fixed point.

Theorem 4.1.4 ***(Leray-Schauder Alternative)***

Let X be a Banach space, C a convex and closed subset in X, U is an open subset of C and $0 \in U$. Suppose that $F : \widetilde{U} \longrightarrow C$ a continuous and compact operator ($F(\widetilde{U})$ is relatively compact of C). Then

(i) F admits a fixed point of $\widetilde{U}$, or

(ii) There exists a $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Theorem 4.1.5 ***(Schaefer)***

Let X be a Banach space and $F : X \longrightarrow X$ a completely continuous operator. If the set

$$\varepsilon = \{u \in X : \lambda F u = u, \lambda \in]0,1[\}$$

is bounded, then F has at least one fixed point.

Theorem 4.1.6 ***(Krasnoselskii)***

Let M be a closed, bounded, convex and non-empty subset of a Banach space X.

Let A and B be two operators such that

(a) $Ax + By \in M, \forall x, y \in M$.

(b) A is compact and continuous.

(c) B is a contracting operator.

Then there exists $z \in M$ such that $z = Az + Bz$.

4.2 Cauchy problem of fractional differential equation

We will study the existence and uniqueness of the solution of a Cauchy problem for fractional dirrential equations (we use the derivative in the sense of Caputo) and we have the problem in the following form

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases}, \tag{4.2.1}$$

where $t \in [0, T], 0 < \alpha \leq 1, u_0 \in \mathbb{R}$, and $f : [0; T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 4.2.1 *Let $0 < \alpha \leq 1$ and let $h : [0; T] \to \mathbb{R}$ a continuous function. A function u is a solution of the Cauchy problem*

$$\begin{cases} {}^C D^\alpha u(t) = h(t), t \in [0, T], 0 < \alpha \leq 1 \\ u(0) = u_0, u_0 \in \mathbb{R} \end{cases}, \tag{4.2.2}$$

if and only if it is the solution of the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds$$