Theorem 4.1.5 ***(Schaefer)***

Let X be a Banach space and $F : X \longrightarrow X$ a completely continuous operator. If the set

$$\varepsilon = \{u \in X : \lambda F u = u, \lambda \in]0,1[\}$$

is bounded, then F has at least one fixed point.

Theorem 4.1.6 ***(Krasnoselskii)***

Let M be a closed, bounded, convex and non-empty subset of a Banach space X.

Let A and B be two operators such that

(a) $Ax + By \in M, \forall x, y \in M$.

(b) A is compact and continuous.

(c) B is a contracting operator.

Then there exists $z \in M$ such that $z = Az + Bz$.

4.2 Cauchy problem of fractional differential equation

We will study the existence and uniqueness of the solution of a Cauchy problem for fractional differential equations (we use the derivative in the sense of Caputo) and we have the problem in the following form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases}, \tag{4.2.1}$$

where $t \in [0, T], 0 < \alpha \leq 1, u_0 \in \mathbb{R}$, and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Lemma 4.2.1 *Let $0 < \alpha \leq 1$ and let $h : [0, T] \rightarrow \mathbb{R}$ a continuous function. A function u is a solution of the Cauchy problem*

$$\begin{cases} {}^{C}D^{\alpha}u(t) = h(t), t \in [0, T], 0 < \alpha \leq 1 \\ u(0) = u_0, u_0 \in \mathbb{R} \end{cases}, \tag{4.2.2}$$

if and only if it is the solution of the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds. \tag{4.2.3}$$

Proof. We apply the operator I^α to equation (4.2.2) we find

$$\begin{aligned} I^\alpha\, {}^C D^\alpha u(t) &= I^\alpha h(t) \\ &\Longrightarrow u(t) + c_0 = I^\alpha h(t) \\ &\Longrightarrow u(t) = I^\alpha h(t) - c_0. \end{aligned}$$

The initial condition gives

$$\begin{aligned} u(0) &= I^\alpha h(0) - c_0 = -c_0 \\ &\Longrightarrow c_0 = -u_0. \end{aligned}$$

So

$$\begin{aligned} u(t) &= I^\alpha h(t) - (-u_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + u_0. \end{aligned}$$

Conversely we have

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= I^\alpha h(t) + u_0. \end{aligned}$$

We apply ${}^C D^\alpha$ to the integral equation (4.2.3)

$$\begin{aligned} {}^C D^\alpha u(t) &= {}^C D^\alpha I^\alpha h(t) + {}^C D^\alpha u_0 \\ &= h(t). \end{aligned}$$

All that remains is to verify that $u(0) = u_0$

$$\begin{aligned} u(0) &= I^\alpha h(0) + u_0 = 0 + u_0 \\ &= u_0. \end{aligned}$$

Then there is a solution to the problem (4.2.2). ■

Theorem 4.2.1 *Let $0 < \alpha \leq 1$ and $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and verifies the following Lipschitz condition*

$$|f(t,u) - f(t,v)| \leq k\,|u - v|\,, \forall t \in [0,T] \text{ and } u, v \in \mathbb{R}.$$

If

$$\frac{kT^\alpha}{\Gamma(\alpha+1)} < 1,$$

then there exists a unique solution to the Cauchy problem (4.2.1).

Proof. We use Banach's fixed point theorem 4.1.2.

We transform the problem (4.2.1) into a fixed point problem (Lemma 4.2.1), by considering the operator

$$\begin{aligned} F : C([0,T],\mathbb{R}) &\longrightarrow C([0,T],\mathbb{R}) \\ u \qquad\qquad &\longrightarrow Fu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, \end{aligned}$$

where $C([0,T],\mathbb{R})$ is the Banach space of continuous functions u defined from $[0;T]$ in $\mathbb{R}$, equipped with the norm

$$\|u\| = \sup_{t\in[0,T]} |u(t)|\,.$$

It is clear that the fixed points of operator F are the solutions of problem (4.2.1). F is well defined, in fact: if $u(t) \in C([0,T],\mathbb{R})$, then $Fu(t) \in C([0,T],\mathbb{R})$.

To show that F admits a fixed point, it suffices to show that F is a contraction, in fact if $u_1, u_2 \in C([0,T],\mathbb{R}), t \in [0,T]$, using the Lipschitiz condition we obtain

$$\begin{aligned} |Fu_1(t) - Fu_2(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,u_1(s)) - f(s,u_2(s))\right) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |f(s,u_1(s)) - f(s,u_2(s))| \, (t-s)^{\alpha-1} ds \\ &\leq \frac{k}{\Gamma(\alpha)} \int_0^t |u_1(s) - u_2(s)| \, (t-s)^{\alpha-1} ds \\ &\leq \frac{k}{\Gamma(\alpha)} |u_1(s) - u_2(s)| \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{kT^\alpha}{\alpha\Gamma(\alpha)} |u_1(s) - u_2(s)|\,. \end{aligned}$$

So

$$\|Fu_1 - Fu_2\|_\infty \leq \frac{kT^\alpha}{\Gamma(\alpha+1)} \|u_1 - u_2\|_\infty\,.$$

By virtue of $\frac{kT^\alpha}{\Gamma(\alpha+1)} < 1$, we can deduce that F is a contraction and according to Banach's theorem, F admits a unique fixed point which is a solution of problem (4.2.1). ■