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Solved exercises

Exercise 1 Suppose that the joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} e^{-\left(\frac{x}{y}+y\right)}y^{-1} & 0 < x, y \\ 0 & \text{otherwise} \end{cases}$$

For $y > 0$, determine :
1- $P(X > 1 \mid Y = y)$;
2- $E[X \mid Y = y]$.
Solution
1- For $y > 0$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{\infty} e^{-\left(\frac{x}{y}+y\right)}y^{-1}dx \\ &= e^{-y} \end{aligned}$$

Hence, for $y > 0$,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \begin{cases} e^{-x/y}y^{-1} & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} P(X > 1 \mid Y = y) &= \int_1^{\infty} f_{X|Y}(x \mid y)dx \\ &= \int_1^{\infty} e^{-x/y}y^{-1}dx \\ &= e^{-1/y} \end{aligned}$$

2-

$$\begin{aligned} E[X \mid Y = y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)dx \\ &= \int_0^{\infty} \frac{x}{y}e^{-x/y}dx \\ &= y \end{aligned}$$

Exercise 2 Let X and Y be continuous random variables having the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 24x(1-x-y), & \text{if } x, y \geq 0, x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

1- Determine the conditional PDF of X given $Y = y$.
2- Determine the conditional PDF of X given $Y = \frac{1}{2}$.
Solution
1- We have the marginal PDf of Y is

$$f_Y(y) = \begin{cases} 4(1-y)^3, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{24x(1-x-y)}{4(1-y)^3}, & \text{if } x,y \geq 0, x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

2-

$$f_{X|Y}(x \mid 0.5) = \frac{f_{X,Y}(x,0.5)}{f_Y\left(\frac{1}{2}\right)} = \begin{cases} \frac{24x(0.5-x)}{4(0.5)^3} = 48x(0.5-x), & \text{if } 0 \leq x \leq 0.5 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 3 Let consider the context of Exercise 3,of the previous chapter, where X gave the amount of gas stocked and Y gave the amount of gas sold at a given.
1- Find the conditional PDF for the amount of gas sold in a given week, when only half of the tank was stocked ; in other word, find the conditional pdf of Y given that $X = 0.5$.
2- Compute the conditional mean value of Y, given that $X = 0.5$

Solution
1- The marginal PDF for X is :

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_0^x 3xdy = 3xy|_0^x = 3x^2, \quad \text{for } 0 \leq x \leq 1$$

Hence, if $X = 0.5$, then $f_X(0.5) = 3(0.5)^2 = 0.75$, and the conditional PDF of Y in this case is

$$f_{Y|X}(y \mid 0.5) = \frac{f(0.5,y)}{f_X(0.5)} = \frac{3(0.5)}{0.75} = 2, \quad \text{for } 0 \leq y \leq 0.5$$

2-

$$\mathrm{E}[Y \mid X = 0.5] = \int_0^{0.5} y f_{Y|X}(y \mid 0.5)dy = 0.25.$$

This means that, when 50% of the tank is stocked, we expect that 25% will be sold.

Exercise 4 Suppose that the joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 3y(x+\frac{1}{4}y) & 0 \leq x,y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

1- Find the conditional PDF of X given $Y = y$, for $y \in (0,1]$.
2- Determine $E[X|Y = y]$, for $y \in (0,1]$.
3- Find $P(X > \frac{1}{2}|Y = 1)$
Solution
1-

$$\begin{aligned} f_Y(y) &= \int_0^1 3y\left(x+\frac{1}{4}y\right)dx = \left[3y\left(\frac{1}{2}x^2+\frac{1}{4}xy\right)\right]_0^1 \\ &= 3y\left(\frac{1}{2}+\frac{1}{4}y\right) = \frac{3}{4}y(2+y). \end{aligned}$$

Therefore for $y \in (0,1]$,

$$\begin{aligned} f_{X|Y}(x|y) &= \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{3y(x+\frac{1}{4}y)}{\frac{3}{4}y(2+y)}, & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{4x+y}{2+y}, & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2-

$$\begin{aligned} E[X|Y=y] &= \int_0^1 x f_{X|Y}(x|y)dx \\ &= \int_0^1 x\frac{4x+y}{2+y}dx = \frac{1}{2+y}\int_0^1 x(4x+y)dx \\ &= \frac{1}{2+y}\left[\frac{4}{3}x^3+\frac{1}{2}x^2y\right]_0^1 = \frac{1}{2+y}\left(\frac{4}{3}+\frac{1}{2}y\right) \\ &= \frac{8+3y}{6(2+y)}. \end{aligned}$$

3-

$$\begin{aligned} P\left(X>\frac{1}{2}\middle|Y=1\right) &= \int_{1/2}^1 f_{X|Y}(x|y=1)dx \\ &= \frac{1}{3}\int_{1/2}^1 (4x+1)dx \\ &= \frac{1}{3}\left[2x^2+x\right]_{1/2}^1 \\ &= \frac{1}{3}\left(2+1-\frac{2}{4}-\frac{1}{2}\right) = \frac{2}{3}. \end{aligned}$$