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Solved Exercises

Exercise 1. Using the Newton-Raphson algorithm, find the square root of 2 in the interval $[1, 2]$ with a precision of $\varepsilon = 10^{-3}$, using $x_0 = 2$ as a starting starting point.

Solution .

We seek the square root of 2 in the interval $[1, 2]$, i.e., we find the root of the following equation

$$x^2 = 2 \Rightarrow f(x) = x^2 - 2 = 0.$$

For all $k \in \mathbb{N}$, we apply the Newton-Raphson iterative scheme :

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, 2, \dots$$

We have $f'(x) = 2x$ and $f''(x) = 2 > 0$, so

$$M_2 = \max_{[1,2]} |f''(x)| = 2 \quad \text{and} \quad m_1 = \min_{[1,2]} |f'(x)| = f'(1) = 2.$$

$$\begin{aligned}
k = 1 : x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} &= 2 - \frac{f(2)}{f'(2)} = 1.5 \\
|\xi - x_1| \leq \frac{M_2}{2m_1}(x_1 - x_0)^2 &= \frac{2}{2 \times 2}(2 - 1.5)^2 = 0.125 \\
k = 2 : x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} &= 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.416 \\
|\xi - x_2| \leq \frac{M_2}{2m_1}(x_2 - x_1)^2 &= \frac{2}{2 \times 2}(1.5 - 1.416)^2 = 0.0035 \\
k = 3 : x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} &= 1.416 - \frac{f(1.416)}{f'(1.416)} = 1.414 \\
|\xi - x_3| \leq \frac{M_2}{2m_1}(x_3 - x_2)^2 &= \frac{2}{2 \times 2}(1.416 - 1.414)^2 = 2 \times 10^{-6} < \varepsilon
\end{aligned}$$

Thus, $x^* \approx 1.414 \pm 2 \times 10^{-6}$ is the approximated square root of 2.

Exercise 2.

a- Give the iterative scheme of the Newton-Raphson algorithm to solve a nonlinear equation $f(x) = 0$.

b- Using the Newton-Raphson algorithm, determine the root in the interval $[0, 1]$ of the equation $x^2 = e^{-2x}$ with a precision of 10^{-3}, starting with an initial point $x_0 = 1$.

Solution .

b- Let's determine the root in the interval $[0, 1]$ of the equation $x^2 = e^{-2x}$ with a precision of 10^{-3} using the Newton-Raphson algorithm.
We have

$$f'(x) = 2x + 2e^{-2x} \quad \text{and} \quad f''(x) = 2 - 4e^{-2x},$$

with

$$M_2 = \max_{[0,1]} |f''(x)| = f''(1) = 1.45 \quad \text{and} \quad m_1 = \min_{[0,1]} |f'(x)| = f'(0.346) = 1.69.$$

Proceeding as in the previous exercise, we get :

$$
\begin{aligned}
k=1: x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} &= 1 - \frac{f(1)}{f'(1)} = 0.6192 \\
|\xi - x_1| &\leq \frac{M_2}{2m_1}(x_1 - x_0)^2 = 0.0624 \\
k=2: x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} &= 0.6192 - \frac{f(0.6192)}{f'(0.6192)} = 0.5677 \\
|\xi - x_2| &\leq \frac{M_2}{2m_1}(x_2 - x_1)^2 = 0.0011 \\
k=3: x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} &= 0.5677 - \frac{f(0.5677)}{f'(0.5677)} = 0.5671 \\
|\xi - x_3| &\leq \frac{M_2}{2m_1}(x_3 - x_2)^2 = 1.55 \times 10^{-7} < \varepsilon
\end{aligned}
$$

Thus, $x^* \approx 0.5671 \pm 1.55 \times 10^{-7}$ is the approximated root.

Exercise 3. Consider the equation $f(x) = 2\tan(x) - x - 1 = 0$ with $x \in [-\pi, \pi]$.

a- Separate analytically the roots of this equation.

b- Calculate the number n of required iterations to approximate this root with a precision of 10^{-3} using the bisection method.

Solution .

a- We have $f(x) = 2\tan(x) - x - 1$, and $f'(x) = \frac{2}{\cos(x)^2} - 1$. The table of variations of f is given as follows :

x	$-\pi$		$-\pi/2$		$\pi/2$		π
$f'(x)$		$+$	‖	$+$	‖	$+$	
$f(x)$	2.14	↗	$+\infty$ ‖ $-\infty$	↗	$+\infty$ ‖ $-\infty$	↗	-4.14

Thus, according to this table, there exists a single root in the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

b- Let's calculate the required number of iterations n :

$$
\begin{aligned}
n &\geq \frac{\ln\left(\frac{b-a}{2\varepsilon}\right)}{\ln 2} \\
&\geq \frac{\ln\left(\frac{\pi}{2\times 10^{-3}}\right)}{\ln 2} \simeq 10.6173
\end{aligned}
$$

Therefore, to reach the root with a precision of 2×10^{-3}, we need at least $n \geq 11$.

Exercice 4 .

a. Approximate the smallest root of the function $f(x) = x^4 - 2x - 4$ with a precision of 5×10^{-3} using Newton-Raphson and Lagrange methods.

b. Compare the two methods and draw a conclusion.

Solution . According to Figure 1, this function $f(x)$ has two roots ; let's find the negative root located in the interval $[-2, -1]$.

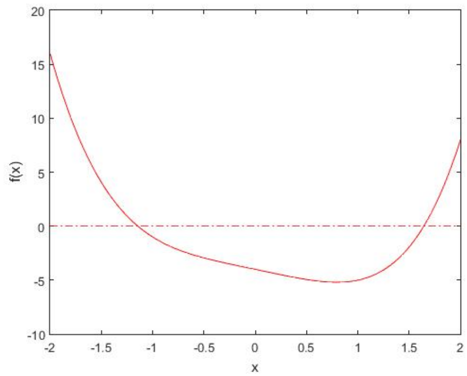

FIGURE 1: Graph of f.

Approximating using Newton-Raphson method :
We have

$$f'(x) = 4x^3 - 2 < 0, \forall x \in [-2,-1],$$
$$f''(x) = 12x^2 > 0, \forall x \in [-2,-1].$$

and

$$M_2 = \max_{[-2,-1]}\{|f''(x)|\} = |f''(-2)| = 48$$
$$m_1 = \min_{[-2,-1]}\{|f'(x)|\} = |f'(-1)| = 6$$

Since $f(-2).f''(-2) > 0$, we take $x_0 = -2$ as a starting point, and for all $k \in \mathbb{N}$, we set

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, 2, \dots.$$

Following the iterative scheme of the Newton-Raphson algorithm, we obtain :

$$k = 1 : x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 - \frac{16}{-34} = -1.53$$
$$|\xi - x_1| \leq \frac{M_2}{2m_1}(x_1 - x_0)^2 = \frac{48}{2 \times 6}(-1.53 + 2)^2 = 0.88$$
$$k = 2 : x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.53 - \frac{-4.53}{-16.32} = -1.25$$
$$|\xi - x_2| \leq \frac{M_2}{2m_1}(x_2 - x_1)^2 = 0.31$$
$$k = 3 : x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.25 - \frac{0.94}{-9.81} = -1.1542$$
$$|\xi - x_3| \leq \frac{M_2}{2m_1}(x_3 - x_2)^2 = 0.03$$
$$k = 4 : x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -1.1542 - \frac{0.083}{-8.15} = -1.144$$
$$|\xi - x_4| \leq \frac{M_2}{2m_1}(x_4 - x_3)^2 = 0.004$$

Hence $\xi = -1.144 \pm 0.004$ is the prescribed solution.
Approximating using Lagrange method :

Since $f(-1).f''(-1) < 0$, we take $x_0 = -1$ as initial point,and for all $n \in \mathbb{N}$, we set

$$x_{n+1} = x_n - f(x_n)\frac{x_n + 2}{f(x_n) - f(-2)}$$

with $M_1 = \max_{[-2,-1]}\{|f'(x)|\} = |f'(-2)| = 34$ and $m_1 = \min_{[-2,-1]}\{|f'(x)|\} = |f'(-1)| = 6$

$$k = 1 : x_1 = x_0 - f(x_0)\frac{x_0 + 2}{f(x_0) - f(-2)} = -1.05$$

$$|x_1 - \xi| \le \frac{M_1 - m_1}{m_1}|x_1 - x_0| = 0.274$$

$$k = 2 : x_2 = x_1 - f(x_1)\frac{x_1 + 2}{f(x_1) - f(-2)} = -1.0941$$

$$|x_2 - \xi| \le \frac{M_1 - m_1}{m_1}|x_2 - x_1| = 0.164$$

$$k = 3 : * x_3 = x_2 - f(x_2)\frac{x_2 + 2}{f(x_2) - f(-2)} = -1.1149$$

$$|x_3 - \xi| \le \frac{M_1 - m_1}{m_1}|x_3 - x_2| = 0.097$$

$$k = 4 : x_4 = x_3 - f(x_3)\frac{x_3 + 2}{f(x_3) - f(-2)} = -1.127$$

$$|x_4 - \xi| \le \frac{M_1 - m_1}{m_1}|x_4 - x_3| = 0.0564$$

$$k = 5 : x_5 = x_4 - f(x_4)\frac{x_4 + 2}{f(x_4) - f(-2)} = -1.1341$$

$$|x_5 - \xi| \le \frac{M_1 - m_1}{m_1}|x_5 - x_4| = 0.033$$

$$k = 6 : x_6 = x_5 - f(x_5)\frac{x_5 + 2}{f(x_5) - f(-2)} = -1.1382$$

$$|x_6 - \xi| \le \frac{M_1 - m_1}{m_1}|x_6 - x_5| = 0.0191$$

$$k = 7 : x_7 = x_6 - f(x_6)\frac{x_6 + 2}{f(x_6) - f(-2)} = -1.1406$$

$$|x_7 - \xi| \le \frac{M_1 - m_1}{m_1}|x_7 - x_6| = 0.011$$

$$k = 8 : x_8 = x_7 - f(x_7)\frac{x_7 + 2}{f(x_7) - f(-2)} = -1.1419$$

$$|x_8 - \xi| \le \frac{M_1 - m_1}{m_1}|x_5 - x_4| = 0.006$$

$$k = 9 : x_9 = x_8 - f(x_8)\frac{x_8 + 2}{f(x_8) - f(-2)} = -1.14275$$

$$|x_9 - \xi| \le \frac{M_1 - m_1}{m_1}|x_9 - x_8| = 0.004$$

Hence$\xi = -1.14275 \pm 0.004$.

To achieve a precision of 5×10^{-3}, it would require 9 iterations using the Lagrange method, whereas the Newton method requires only 4 iterations. The Newton method converges faster than the Lagrange method.

Exercice 5. We consider the equation $f(x) = 0$, with $f(x) = \ln(x) - x + 2$.

1.a. Write the equation $f(x) = 0$ in the form $f_1(x) = f_2(x)$ with $f_1(x) = \ln(x)$.

b. Plot the graphs of f_1 and f_2. What can be said about this equation ?

2.a. Perform 4 iterations of the bisection method to approximate the solution in the interval $[3, 4]$. At which iteration we obtain the best result ? Justify and conclude.
b. Determine the number of n of required iterations to achieve a precision of 10^{-4}.
c. Give an estimate of the error after 25 iterations.
3. Approximate the root with a precision of 10^{-4} using the Newton-Raphson method, starting from $x_0 = 3$.
4. Compare the two methods and draw a conclusion.

Solution .

1-a.

$$\begin{aligned} f(x) = 0 &\Leftrightarrow \ln(x) - x + 2 = 0 \\ &\Leftrightarrow \ln(x) = x - 2 \\ &\Leftrightarrow f_1(x) = f_2(x) \text{ avec } f_1(x) = \ln(x) \text{ et } f_2(x) = x - 2 \end{aligned}$$

1-b. According to Figure 2, the graphs of f_1 and f_2 have two intersection points, so this equation has two roots $\xi_1 \in]0, 1[$ and $\xi_2 \in]3, 4[$.

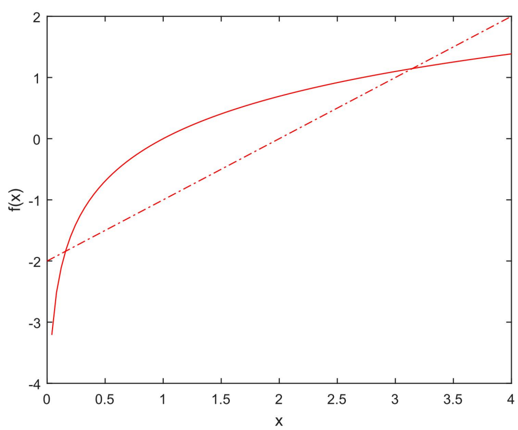

FIGURE 2: Graphical separation of the roots.

2-a. Proceeding as in examples 1 and 2, we obtain $x_1 = 3.5$, $x_2 = 3.25$, $x_3 = 3.125$, and $x_4 = 3.1875$, with x_3 the best result obtained since $f(x_3) \min\{f(x_i), i = 1, 2, 3, 4\}$. We conclude that, although the convergence of the bisection method towards the root is guaranteed, it is not monotonic.
2-b.

$$\begin{aligned} n &\geq \frac{\ln(\frac{b-a}{2\varepsilon})}{\ln 2} \\ &\geq \frac{\ln(\frac{1}{10^{-4}})}{\ln 2} \simeq 13.29\,, \end{aligned}$$

hence $n \geq 14$.
2-c

$$\begin{aligned} |x_n - \xi| &\leq \frac{b-a}{2^{n+1}} \\ &= \frac{4-3}{2^{26}} = 1.4901 \times 10^{-8} \end{aligned}$$

3. By applying the Newton algorithm starting from the point $x_0 = 3$, and after 3 iterations, the algorithm reaches the root within 10^{-4}. The generated points are $x_1 = 3.1479$, $x_2 = 3.1462$, and $x_3 = 3.1462$.
4. To achieve a precision of 10^{-4}, it would require 14 iterations using the bisection method, whereas the Newton method only requires 3 iterations. The Newton-Raphson method converges quicker than the bisection method.